Dynamic Games of Complete Information

Dynamic Games of Complete and Perfect Information
Outline of dynamic games of complete information

- Dynamic games of complete information
- Extensive-form representation
- Dynamic games of complete and perfect information
- Game tree
- Subgame-perfect Nash equilibrium
- Backward induction
- Applications
- Dynamic games of complete and imperfect information
- More applications
- Repeated games
Today’s Agenda

- Review of previous class
- Subgame-perfect Nash equilibrium
- Backward induction
- Stackelberg’s model of duopoly (2.1.B of Gibbons, 6.2 of Osborne)
- Sequential-move Bertrand model of duopoly (differentiated products)
Dynamic games of complete and perfect information

Perfect information

- All previous moves are observed before the next move is chosen.
- A player knows **Who** has moved **What** before she makes a decision.
A strategy for a player is a complete plan of actions.

It specifies a feasible action for the player in every contingency in which the player might be called on to act.

It specifies what the player does at each of her nodes.

Player 1’s payoff is -1 and player 2’s payoff is 1 if player 1 plays H and player 2 plays HT.
Nash equilibrium in a dynamic game

- We can also use normal-form to represent a dynamic game.
- The set of Nash equilibria in a dynamic game of complete information is the set of Nash equilibria of its normal-form.
- How to find the Nash equilibria in a dynamic game of complete information:
  - Construct the normal-form of the dynamic game of complete information.
  - Find the Nash equilibria in the normal-form.
Subgame-perfect Nash equilibrium

- A Nash equilibrium of a dynamic game is subgame-perfect if the strategies of the Nash equilibrium constitute or induce a Nash equilibrium in every subgame of the game.

- Subgame-perfect Nash equilibrium is a Nash equilibrium.
A subgame of a game tree begins at a nonterminal node and includes all the nodes and edges following the nonterminal node.

A subgame beginning at a nonterminal node $x$ can be obtained as follows:

- remove the edge connecting $x$ and its predecessor
- the connected part containing $x$ is the subgame

A subgame

Game Theory – Fabrice Valognes -- Chp 15
Existence of subgame-perfect Nash equilibrium

- Every finite dynamic game of complete and perfect information has a subgame-perfect Nash equilibrium that can be found by backward induction.
Backward induction: illustration

- Subgame-perfect Nash equilibrium \((C, EH)\).
  - player 1 plays \(C\);
  - player 2 plays \(E\) if player 1 plays \(C\), plays \(H\) if player 1 plays \(D\).
Subgame-perfect Nash equilibrium (D, FHK).

- player 1 plays D
- player 2 plays F if player 1 plays C, plays H if player 1 plays D, plays K if player 1 plays E.
Multiple subgame-perfect Nash equilibria

- Subgame-perfect Nash equilibrium \((E, FHK)\).
  - player 1 plays \(E\);
  - player 2 plays \(F\) if player 1 plays \(C\), plays \(H\) if player 1 plays \(D\), plays \(K\) if player 1 plays \(E\).
Multiple subgame-perfect Nash equilibria

- Subgame-perfect Nash equilibrium \((D, FIK)\).
  - player 1 plays \(D\);
  - player 2 plays \(F\) if player 1 plays \(C\), plays \(I\) if player 1 plays \(D\), plays \(K\) if player 1 plays \(E\).
Stackelberg model of duopoly

- A homogeneous product is produced by only two firms: firm 1 and firm 2. The quantities are denoted by $q_1$ and $q_2$, respectively.
- The timing of this game is as follows:
  - Firm 1 chooses a quantity $q_1 \geq 0$.
  - Firm 2 observes $q_1$ and then chooses a quantity $q_2 \geq 0$.
- The market price is $P(Q) = a - Q$, where $a$ is a constant number and $Q = q_1 + q_2$.
- The cost to firm $i$ of producing quantity $q_i$ is $C_i(q_i) = cq_i$.
- Payoff functions:
  \[ u_1(q_1, q_2) = q_1(a - (q_1 + q_2) - c) \]
  \[ u_2(q_1, q_2) = q_2(a - (q_1 + q_2) - c) \]
Stackelberg model of duopoly

- Find the subgame-perfect Nash equilibrium by backward induction
  - We first solve firm 2’s problem for any $q_1 \geq 0$ to get firm 2’s best response to $q_1$. That is, we first solve all the subgames beginning at firm 2.
  - Then we solve firm 1’s problem. That is, solve the subgame beginning at firm 1.
Stackelberg model of duopoly

- Solve firm 2’s problem for any $q_1 \geq 0$ to get firm 2’s best response to $q_1$.
  - Max $u_2(q_1, q_2) = q_2(a - (q_1 + q_2) - c)$ subject to $0 \leq q_2 \leq +\infty$
  - FOC: $a - 2q_2 - q_1 - c = 0$

- Firm 2’s best response,
  - $R_2(q_1) = (a - q_1 - c)/2$ if $q_1 \leq a - c$
  - $= 0$ if $q_1 > a - c$

Note: Osborne used $b_2(q_1)$ instead of $R_2(q_1)$
Stackelberg model of duopoly

Solve firm 1’s problem. Note firm 1 can also solve firm 2’s problem. That is, firm 1 knows firm 2’s best response to any $q_1$. Hence, firm 1’s problem is

Max $u_1(q_1, R_2(q_1)) = q_1(a - (q_1 + R_2(q_1)) - c)$
subject to $0 \leq q_1 \leq +\infty$

That is,
Max $u_1(q_1, R_2(q_1)) = q_1(a - q_1 - c)/2$
subject to $0 \leq q_1 \leq +\infty$

FOC: $(a - 2q_1 - c)/2 = 0$
$q_1 = (a - c)/2$
Stackelberg model of duopoly

- **Subgame-perfect Nash equilibrium**
  - \(( (a - c)/2, R_2(q_1) )\), where
    \[
    R_2(q_1) = \begin{cases} 
    (a - q_1 - c)/2 & \text{if } q_1 \leq a - c \\
    0 & \text{if } q_1 > a - c
    \end{cases}
    \]
  - That is, firm 1 chooses a quantity \((a - c)/2\), firm 2 chooses a quantity \(R_2(q_1)\) if firm 1 chooses a quantity \(q_1\).
  - The *backward induction outcome* is \(( (a - c)/2, (a - c)/4 )\).
  - Firm 1 chooses a quantity \((a - c)/2\), firm 2 chooses a quantity \((a - c)/4\).
Stackelberg model of duopoly

- Firm 1 produces
  \[ q_1 = \frac{(a - c)}{2} \]
  and its profit
  \[ q_1(a-(q_1 + q_2)-c) = \frac{(a-c)^2}{8} \]

- Firm 2 produces
  \[ q_2 = \frac{(a - c)}{4} \]
  and its profit
  \[ q_2(a-(q_1 + q_2)-c) = \frac{(a-c)^2}{16} \]

- The aggregate quantity is \( 3(a - c)/4 \).
Cournot model of duopoly

- Firm 1 produces
  \[ q_1 = \frac{a - c}{3} \] and its profit
  \[ q_1(a-(q_1+q_2)-c) = \frac{(a-c)^2}{9} \]

- Firm 2 produces
  \[ q_2 = \frac{a - c}{3} \] and its profit
  \[ q_2(a-(q_1+q_2)-c) = \frac{(a-c)^2}{9} \]

- The aggregate quantity is \( 2(a - c)/3 \).
Monopoly

- Suppose that only one firm, a monopoly, produces the product. The monopoly solves the following problem to determine the quantity $q_m$.

- Max $q_m (a - q_m - c)$
- subject to $0 \leq q_m \leq +\infty$

  FOC: $a - 2q_m - c = 0$
  $q_m = (a - c)/2$

- Monopoly produces $q_m = (a - c)/2$ and its profit
  $q_m(a - q_m - c) = (a - c)^2/4$
Sequential-move Bertrand model of duopoly (differentiated products)

- Two firms: firm 1 and firm 2.
- Each firm chooses the price for its product. The prices are denoted by $p_1$ and $p_2$, respectively.
- The timing of this game as follows.
  - Firm 1 chooses a price $p_1 \geq 0$.
  - Firm 2 observes $p_1$ and then chooses a price $p_2 \geq 0$.
- The quantity that consumers demand from firm 1:
  $$q_1(p_1, p_2) = a - p_1 + bp_2.$$ 
- The quantity that consumers demand from firm 2:
  $$q_2(p_1, p_2) = a - p_2 + bp_1.$$ 
- The cost to firm $i$ of producing quantity $q_i$ is $C_i(q_i) = cq_i$. 
Sequential-move Bertrand model of duopoly (differentiated products)

- Solve firm 2’s problem for any $p_1 \geq 0$ to get firm 2’s best response to $p_1$.
  - Max $u_2(p_1, p_2) = (a - p_2 + bp_1)(p_2 - c)$ subject to $0 \leq p_2 \leq +\infty$
  - FOC: $a + c - 2p_2 + bp_1 = 0$
    - $p_2 = (a + c + bp_1)/2$
  - Firm 2’s best response,
    - $R_2(p_1) = (a + c + bp_1)/2$
Sequential-move Bertrand model of duopoly (differentiated products)

- Solve firm 1’s problem. Note firm 1 can also solve firm 2’s problem. Firm 1 knows firm 2’s best response to $p_1$. Hence, firm 1’s problem is

  $\max u_1(p_1, R_2(p_1)) = (a - p_1 + b \times R_2(p_1))(p_1 - c)$
  subject to $0 \leq p_1 \leq +\infty$

  That is,
  $\max u_1(p_1, R_2(p_1)) = (a - p_1 + b \times (a + c + bp_1)/2)(p_1 - c)$
  subject to $0 \leq p_1 \leq +\infty$

  - FOC: $a - p_1 + b \times (a + c + bp_1)/2 + (-1 + b^2/2)(p_1 - c) = 0$
  $p_1 = (a + c + (ab + bc - b^2c)/2)/(2 - b^2)$
Sequential-move Bertrand model of duopoly (differentiated products)

- **Subgame-perfect Nash equilibrium**
  
  - \[(\frac{(a+c+(ab+bc-b^2c)/2)}{2-b^2}), \ R_2(p_1)\],
  
  where \( R_2(p_1) = (a + c + bp_1)/2 \)

  - Firm 1 chooses a price \( (a+c+(ab+bc-b^2c)/2)/(2-b^2) \),
  
  firm 2 chooses a price \( R_2(p_1) \) if firm 1 chooses a price \( p_1 \).
Summary

- Subgame perfect Nash equilibrium
- Backward induction
- Stackelberg model of duopoly
- Sequential-move Bertrand model of duopoly (differentiated products)

Next time
- Dynamic games of complete and imperfect information

Reading lists
- Sec 2.2, 2.4 of Gibbons