Static (or Simultaneous-Move) Games of Complete Information

Introduction to Games
Normal (or Strategic) Form Representation
Outline of Static Games of Complete Information

- Introduction to games
- Normal-form (or strategic-form) representation
- Iterated elimination of strictly dominated strategies
- Nash equilibrium
- Review of concave functions, optimization
- Applications of Nash equilibrium
- Mixed strategy Nash equilibrium
Agenda

- What is game theory
- Examples
  - Prisoner’s dilemma
  - The battle of the sexes
  - Matching pennies
- Static (or simultaneous-move) games of complete information
- Normal-form or strategic-form representation
What is game theory?

- We focus on games where:
  - There are at least two rational players
  - Each player has more than one choices
  - The outcome depends on the strategies chosen by all players; there is strategic interaction

Example: Six people go to a restaurant.
- Each person pays his/her own meal – a simple decision problem
- Before the meal, every person agrees to split the bill evenly among them – a game
What is game theory?

- **Game theory** is a formal way to analyze strategic interaction among a group of rational players (or agents) who behave strategically.

- Game theory has applications:
  - Economics
  - Politics
  - etc.
Classic Example: Prisoners’ Dilemma

- Two suspects **held in separate cells** are charged with a major crime. However, there is not enough evidence.
- Both suspects are told the following policy:
  - If neither confesses then both will be convicted of a minor offense and sentenced to one month in jail.
  - If both confess then both will be sentenced to jail for six months.
  - If one confesses but the other does not, then the confessor will be released but the other will be sentenced to jail for nine months.

<table>
<thead>
<tr>
<th></th>
<th>Confess</th>
<th>Mum</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Prisoner 1</strong></td>
<td>0 ⏞ -9</td>
<td>-1 ⏞ -1</td>
</tr>
<tr>
<td><strong>Prisoner 2</strong></td>
<td>-6 ⏞ -6</td>
<td>-9 ⏞ 0</td>
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Example: The battle of the sexes

- At the separate workplaces, Chris and Pat must choose to attend either an opera or a prize fight in the evening.

- Both Chris and Pat know the following:
  - Both would like to spend the evening together.
  - But Chris prefers the opera.
  - Pat prefers the prize fight.
Example: Matching pennies

- Each of the two players has a penny.
- Two players must **simultaneously** choose whether to show the Head or the Tail.
- Both players know the following rules:
  - If two pennies match (both heads or both tails) then player 2 wins player 1’s penny.
  - Otherwise, player 1 wins player 2’s penny.

<table>
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<tr>
<th>Player 1</th>
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<tbody>
<tr>
<td>Head</td>
<td>-1, 1</td>
</tr>
<tr>
<td>Tail</td>
<td>1, -1</td>
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</table>
Static (or simultaneous-move) games of complete information

A static (or simultaneous-move) game consists of:

- A set of players (at least two players)
- For each player, a set of strategies/actions
- Payoffs received by each player for the combinations of the strategies, or for each player, preferences over the combinations of the strategies

\[
\{\text{Player 1, Player 2, ... Player } n\} \\
S_1 \ S_2 \ ... \ S_n \\
u_i(s_1, s_2, ...s_n), \text{ for all } s_1 \in S_1, s_2 \in S_2, ... \ s_n \in S_n.
\]
Static (or simultaneous-move) games of complete information

- Simultaneous-move
  - Each player chooses his/her strategy without knowledge of others’ choices.

- Complete information
  - Each player’s strategies and payoff function are common knowledge among all the players.

- Assumptions on the players
  - Rationality
    - Players aim to maximize their payoffs
    - Players are perfect calculators
  - Each player knows that other players are rational
Static (or simultaneous-move) games of complete information

- The players cooperate?
  - No. Only noncooperative games

- The timing
  - Each player $i$ chooses his/her strategy $s_i$ without knowledge of others’ choices.
  - Then each player $i$ receives his/her payoff $u_i(s_1, s_2, ..., s_n)$.
  - The game ends.
Definition: normal-form or strategic-form representation

The normal-form (or strategic-form) representation of a game $G$ specifies:

- A finite set of players $\{1, 2, ..., n\}$,
- players’ strategy spaces $S_1$ $S_2$ $...$ $S_n$ and
- their payoff functions $u_1$ $u_2$ $...$ $u_n$

where $u_i : S_1 \times S_2 \times ... \times S_n \rightarrow R$. 

Normal-form representation: 2-player game

- Bi-matrix representation
  - 2 players: Player 1 and Player 2
  - Each player has a finite number of strategies
- Example:
  \[ S_1 = \{ s_{11}, s_{12}, s_{13} \} \quad S_2 = \{ s_{21}, s_{22} \} \]

<table>
<thead>
<tr>
<th>Player 1</th>
<th>( \text{s}_{21} )</th>
<th>( \text{s}_{22} )</th>
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<tr>
<td>( \text{s}_{11} )</td>
<td>( u_1(s_{11}, s_{21}) ), ( u_2(s_{11}, s_{21}) )</td>
<td>( u_1(s_{11}, s_{22}) ), ( u_2(s_{11}, s_{22}) )</td>
</tr>
<tr>
<td>( \text{s}_{12} )</td>
<td>( u_1(s_{12}, s_{21}) ), ( u_2(s_{12}, s_{21}) )</td>
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Classic example: Prisoners’ Dilemma: normal-form representation

- **Set of players:** \{Prisoner 1, Prisoner 2\}
- **Sets of strategies:** \( S_1 = S_2 = \{\text{Mum, Confess}\} \)
- **Payoff functions:**

\[
\begin{align*}
  u_1(\text{Mum}, \text{Mum}) &= -1, \\
  u_1(\text{Mum}, \text{Confess}) &= -9, \\
  u_1(\text{Confess}, \text{Mum}) &= 0, \\
  u_1(\text{Confess}, \text{Confess}) &= -6; \\
  u_2(\text{Mum}, \text{Mum}) &= -1, \\
  u_2(\text{Mum}, \text{Confess}) &= 0, \\
  u_2(\text{Confess}, \text{Mum}) &= -9, \\
  u_2(\text{Confess}, \text{Confess}) &= -6
\end{align*}
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Example: The battle of the sexes

- Normal (or strategic) form representation:
  - Set of players: \{Chris, Pat\} (=\{Player 1, Player 2\})
  - Sets of strategies: \(S_1 = S_2 = \{\text{Opera}, \text{Prize Fight}\}\)
  - Payoff functions:
    \[
    u_1(O, O)=2, \ u_1(O, F)=0, \ u_1(F, O)=0, \ u_1(F, F)=1;
    u_2(O, O)=1, \ u_2(O, F)=0, \ u_2(F, O)=0, \ u_2(F, F)=2
    \]
Example: Matching pennies

- **Normal (or strategic) form representation:**
  - Set of players: \{Player 1, Player 2\}
  - Sets of strategies: \( S_1 = S_2 = \{ \text{Head, Tail} \} \)
  - Payoff functions:
    - \( u_1(H, H) = -1, \ u_1(H, T) = 1, \ u_1(T, H) = 1, \ u_1(H, T) = -1 \)
    - \( u_2(H, H) = 1, \ u_2(H, T) = -1, \ u_2(T, H) = -1, \ u_2(T, T) = 1 \)

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Example: Tourists & Natives

- Only two bars (bar 1, bar 2) in a city
- Can charge price of $2, $4, or $5
- 6000 tourists pick a bar randomly
- 4000 natives select the lowest price bar

Example 1: Both charge $2
  - each gets 5,000 customers and $10,000

Example 2: Bar 1 charges $4, Bar 2 charges $5
  - Bar 1 gets 3000+4000=7,000 customers and $28,000
  - Bar 2 gets 3000 customers and $15,000
Example: Cournot model of duopoly

- A product is produced by only two firms: firm 1 and firm 2. The quantities are denoted by $q_1$ and $q_2$, respectively. Each firm chooses the quantity without knowing the other firm has chosen.
- The market price is $P(Q)=a-Q$, where $Q=q_1+q_2$.
- The cost to firm $i$ of producing quantity $q_i$ is $C_i(q_i)=cq_i$.

The normal-form representation:
- Set of players: \{Firm 1, Firm 2\}
- Sets of strategies: $S_1=[0, +\infty)$, $S_2=[0, +\infty)$
- Payoff functions:
  
  $u_1(q_1, q_2)=q_1(a-(q_1+q_2)-c)$, $u_2(q_1, q_2)=q_2(a-(q_1+q_2)-c)$
One More Example

- Each of $n$ players selects a number between 0 and 100 simultaneously. Let $x_i$ denote the number selected by player $i$.
- Let $y$ denote the average of these numbers
- Player $i$’s payoff $= x_i - 3y/5$
- The normal-form representation:
Solving Prisoners’ Dilemma

- Confess always does better whatever the other player chooses
- Dominated strategy
  - There exists another strategy which always does better regardless of other players’ choices

\[
\begin{array}{c|cc}
\text{Prisoner 1} & \text{Mum} & \text{Confess} \\
\hline
\text{Mum} & (-1, -1) & (-9, 0) \\
\text{Confess} & (0, -9) & (-6, -6) \\
\end{array}
\]
Definition: strictly dominated strategy

In the normal-form game \( \{ S_1, S_2, ..., S_n, u_1, u_2, ..., u_n \} \), let \( s_i', s_i'' \in S_i \) be feasible strategies for player \( i \). Strategy \( s_i' \) is strictly dominated by strategy \( s_i'' \) if

\[
u_i(s_1, s_2, ..., s_{i-1}, s_i', s_{i+1}, ..., s_n) < u_i(s_1, s_2, ..., s_{i-1}, s_i'', s_{i+1}, ..., s_n)\]

for all \( s_1 \in S_1, s_2 \in S_2, ..., s_{i-1} \in S_{i-1}, s_{i+1} \in S_{i+1}, ..., s_n \in S_n \).

\[
\begin{array}{cc|cc}
\text{Prisoner 1} & \text{Mum} & \text{Confess} \\
\hline
\text{Mum} & -1, -1 & -9, 0 \\
\text{Confess} & 0, -9 & -6, -6 \\
\end{array}
\]

\( s_i'' \) is strictly better than \( s_i' \) regardless of other players’ choices.
Summary

- Static (or simultaneous-move) games of complete information
- Normal-form or strategic-form representation

Next time
- Dominated strategies
- Iterated elimination of strictly dominated strategies
- Nash equilibrium

Reading lists